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Derivation part three (third class) of Leibniz algebras

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A B S T R A C T

In this paper, dimension the derivations of a third class of Leibniz algebras are discussed and studied. For $L \in TLb_n$, we have $n+1 \leqslant dimDer(L) \leqslant 2n-1$ are deduced.

1. Introduction

In 1993 Loday discovered a generalization of Lie algebra called a (left) Liebinz algebra, where every left multiplication operator is a derivation.There is the concept of right Leibniz algebra, where every right multiplication operator also is a derivation([1, 2]).The anti-symmetry condition of the Lie algebra ([u,u]=0) not necessary to be hold in a Leibniz algebra, hence Leibniz algebra is a non commutative analogue of the Lie algebra. Derivations of low-dimensional Leibniz algebras were studied by Rakhimov and AL-Nashri up to dimension eight([3-6]). Also, they studied general derivation for tow classes (First and Second) of Filiform Leibniz algebras ([7, 8]). In this work we describe general the derivations of any algebra from third class of filiform Leibniz algebras ($L \in TLb_n$).The outline of this paper is as follows: in Section 2 we present some preliminary results on the Leibniz algebras, basic definitions and properties are given. In Section 3 we study dimension of the derivation algebra in third classes as following $TLb_n = \cup_{k=1}^{n-1} T_{2n-k}$, where T_k is a sub classes of TLb_n, with the derivation algebras of dimensions k. We study only one subclass from each this series, and the other cases are similar.

2. Preliminary Results

In this section we provide some basic definitions and properties of Leibniz algebras, and focus on so-called filiform Leibniz algebras.

Definition 2.1 (see[9]) An algebra L over a field K is called a Leibniz algebra if its bilinear operation $[.,.]$ satisfies the following Leibniz identity:

$$[x,[y,z]] = [[x,y],z] - [[x,z],y], \ \forall \ x,y,z \in L.$$

From now onwards, all algebras are assumed to be over the fields of complex numbers $\mathbb{C}$. Throughout the paper, we denote by L the Leibniz algebra, and let us use the following sequence: $L^1 = L, L^{k+1} = [L^k, L], k \geqslant 1$.

Definition 2.2 (see[10]) A Leibniz algebra L is nilpotent if there exists $s \in \mathbb{N}$ such that $L^1 \supset L^2 \supset \cdots \supset L^s = 0$.

Definition 2.3 (see[11]) A Leibniz algebra L is filiform if $dim(L^i) = n - i$, where $n = dimL$ and $2 \leqslant i \leqslant n$.

Definition 2.4 (see[12]) A $\mathbb{C}$-linear transformation d of a Leibniz algebra L is called a derivation of L if

$$d([x,y]) = [d(x),y] + [x,d(y)] \quad \forall \ x,y \in L.$$

The set of all derivations of an algebra L is denoted by $Der(L)$. We also, denote by $Leib_n$ the set of all (n+1)-dimensional filiform Leibniz algebras.

Theorem 2.1 Any $(n+1)$-dimensional complex Filiform Leibniz algebra L admits a basis $\{e_0, e_1, \cdots, e_n\}$ called adapted, such that the table of multiplication of L has the following form, where non defined products are zero:

$$TLb_n = \begin{cases} [e_i,e_1] = e_{i+1}, & 2 \leqslant i \leqslant n-1, \\ [e_1,e_i] = -e_{i+1}, & 3 \leqslant i \leqslant n-1, \\ [e_1,e_1] = \theta_1 e_n, & \\ [e_1,e_2] = -e_3 + \theta_2 e_n, & \\ [e_2,e_2] = \theta_3 e_n, & \\ [e_i,e_j] = -[e_j,e_i] = \beta_1 e_{i+j+1} + \beta_2 e_{i+j+2} + \cdots + \beta_{n-i-j} e_n, & 2 \leqslant i \leqslant n-2, \\ & 2 \leqslant j \leqslant n-i, \\ [e_i,e_{n+1-i}] = -[e_{n+1-i},e_i] = \alpha(-1)^{i+1} e_n, & 2 \leqslant i \leqslant n-1. \end{cases}$$

algebras from TLb_n and denoted by $L(\theta_1, \theta_2, \theta_3, \alpha, \dots, \beta_i)$ where $i = 0,1,2 \dots n-1$.

Theorem 2.2 describes the derivation algebras of elements from TLb_n.

Theorem 2.2 If $L \in TLb_n$ then

$$dimDer(L) = \begin{cases} 2n-1 & \begin{cases} \rhd\ if \ \{\theta_1 = 0, \theta_2 = 0, \theta_3 = 0, \alpha = 0, \beta_i \neq 0, 0 \leqslant i \leqslant n-1 \\ \rhd\ if \ \{\theta_1 \neq 0, \theta_2 = 0, \theta_3 = 0, \alpha = 0, \beta_i = 0, 0 \leqslant i \leqslant n-1 \end{cases} \\ 2n-2 & \begin{cases} \bullet\ if \ \{\theta_1 \neq 0, \theta_2 = 0, \theta_3 \neq 0, \alpha = 0, \beta_i = 0, \quad 0 \leqslant i \leqslant n-1 \\ \bullet\ if \ \{\theta_1 \neq 0, \theta_2 = 0, \theta_3 \neq 0, \alpha \neq 0, \beta_i = 0, \quad 0 \leqslant i \leqslant n-1 \end{cases} \\ 2n-3 & \{ \rhd\ if \ \{\theta_1 = 0, \theta_2 = 0, \theta_3 = 0, \alpha \neq 0, \beta_i \neq 0, \quad 0 \leqslant i \leqslant n-1 \\ 2n-4 & \{ \bullet\ if \ \{\theta_1 \neq 0, \theta_2 \neq 0, \theta_3 \neq 0, \alpha \neq 0, \beta_i = 0, \quad 0 \leqslant i \leqslant n-1 \\ \vdots & \\ n+2 & \begin{cases} \rhd\ if \ \{\theta_1 = 0, \theta_2 \neq 0, \theta_3 \neq 0, \alpha = 0, \beta_i \neq 0, 0 \leqslant i \leqslant n-1 \\ \rhd\ if \ \{\theta_1 = 0, \theta_2 \neq 0, \theta_3 = 0, \alpha = 0, \beta_i \neq 0, 0 \leqslant i \leqslant n-1 \end{cases} \\ n+1 & \begin{cases} \bullet\ if \ \{\theta_1 \neq 0, \theta_2 \neq 0, \theta_3 \neq 0, \alpha \neq 0, \beta_i \neq 0, 0 \leqslant i \leqslant n-1 \\ \bullet\ if \ \{\theta_1 \neq 0, \theta_2 \neq 0, \theta_3 \neq 0, \alpha = 0, \beta_i \neq 0, 0 \leqslant i \leqslant n-1 \end{cases} \end{cases}$$

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Proof of Theorem 2.2. We limit our proof to the case where $\theta_1 \neq 0, \theta_2 \neq 0, \theta_3 \neq 0, \alpha \neq 0$ and $\beta_i \neq 0, 0 \leqslant i \leqslant n-1$. The other cases are treated using the same strategy. Assume that $d(e_j) = \sum_{i=1}^{n} d_i^j e_i$, where $j = 1,2$. Since, $[e_1, e_1] = \theta_1 e_n$, we have

$$\begin{aligned} d(e_n) &= \frac{1}{\theta_1}([d(e_1), e_1] + [e_1, d(e_1)]) \\ &= \frac{1}{\theta_1}([\textstyle\sum_{i=1}^{n} d_i^1 e_i, e_1] + [e_1, \sum_{i=1}^{n} d_i^1 e_i]) \\ &= \frac{1}{\theta_1}(d_1^1[e_1,e_1] + d_2^1[e_2,e_1] + [\textstyle\sum_{i=3}^{n} d_i^1 e_i, e_1] + d_1^1[e_1,e_1] + d_2^1[e_1,e_2] + [e_1, \sum_{i=3}^{n} d_i^1 e_i]) \\ &= \frac{1}{\theta_1}(d_1^1[e_1,e_1] + d_2^1[e_2,e_1] + d_1^1[e_1,e_1] + d_2^1[e_1,e_2]) \\ &= \frac{1}{\theta_1}(2d_1^1[e_1,e_1] + d_2^1([e_2,e_1] + [e_1,e_2])) \\ &= \frac{1}{\theta_1}(2d_1^1\theta_1 + d_2^1\theta_2)e_n \end{aligned}$$

That is

$$d(e_n) = (2d_1^1 + d_2^1 \frac{\theta_2}{\theta_1})e_n \,. \qquad (1)$$

Since $[e_2, e_1] = e_3$ then

$$\begin{aligned} d(e_3) &= [d(e_2), e_1] + [e_2, d(e_1)] \\ &= [\textstyle\sum_{i=2}^{n} d_i^2 e_i, e_1] + [e_2, \sum_{i=1}^{n} d_i^1 e_i] \\ &= d_2^2[e_2,e_1] + [\textstyle\sum_{i=3}^{n} d_i^2 e_i, e_1] + d_1^1[e_2,e_1] + d_2^1[e_2,e_2] + [e_2, \sum_{i=3}^{n} d_i^1 e_i] \\ &= (d_2^2 + d_1^1)[e_2,e_1] + \textstyle\sum_{i=3}^{n-1} d_i^2 e_{i+1} + d_2^1[e_2,e_2] + \sum_{i=3}^{n-2} d_i^1[e_2,e_i] + d_{n-1}^1[e_2,e_{n-1}] + d_n^1[e_2,e_n] \end{aligned}$$

Now we make use Theorem 2.1 accordingly

$[e_2, e_i] \in$ span $(e_{i+2}, e_{i+3}, \dots, e_n)$ then $\exists\ \beta_{k-i-2}^2 \in \mathbb{C}, k = i+2, i+3, \dots, n$, such that $[e_2, e_i] = \sum_{k=i+2}^{n} \beta_{k-i-2}^2 e_k$. Hence

$$\begin{aligned} \textstyle\sum_{i=3}^{n-2} d_i^1[e_2,e_i] &= \textstyle\sum_{i=3}^{n-2} d_i^1 \sum_{k=i+2}^{n} \beta_{k-i-2}^2 e_k \\ &= \textstyle\sum_{i=3}^{n-2} \sum_{k=i+2}^{n} d_i^1 \beta_{k-i-2}^2 e_k \end{aligned}$$

To invert the sums of double summations, we use

$$\begin{cases} 3 \leqslant i \leqslant n-2 \\ i+2 \leqslant k \leqslant n \end{cases} \Leftrightarrow \begin{cases} 5 \leqslant k \leqslant n \\ 3 \leqslant i \leqslant k-2 \end{cases} \qquad (2)$$

Now, using (2) we get:

$$\textstyle\sum_{i=3}^{n-2} d_i^1[e_2,e_i] = \sum_{k=5}^{n} (\sum_{i=3}^{k-2} d_i^1 \beta_{k-i-2}^2) e_k$$

Due to Theorem 2.1 we have:

$$\begin{cases} [e_2, e_n] = 0, \\ [e_2, e_{n-1}] = (-1)^3 \alpha e_n = -\alpha e_n, \\ [e_2, e_1] = e_3 \\ [e_2, e_2] = \theta_3 e_n. \end{cases}$$

summing up

$$\begin{aligned} d(e_3) &= (d_2^2 + d_1^1)e_3 + d_2^1\theta_3 e_n + \sum_{i=4}^{n-1} d_{i-1}^2 e_i + d_{n-1}^2 e_n \\ &\quad + \sum_{k=5}^{n} (\sum_{i=3}^{k-2} d_i^1 \beta_{k-i-2}^2) e_k - d_{n-1}^1 \alpha e_n \\ &= (d_2^2 + d_1^1)e_3 + \sum_{i=4}^{n-1} d_{i-1}^2 e_i + \sum_{k=5}^{n-1} (\sum_{i=3}^{k-2} d_i^1 \beta_{k-i-2}^2) e_k \\ &\quad +(d_{n-1}^2 + d_2^1\theta_3 - d_{n-1}^1\alpha + \sum_{i=3}^{n-2} d_i^1 \beta_{n-i-2}^2) e_n \\ &= (d_2^2 + d_1^1)e_3 + d_3^2 e_4 + \sum_{k=5}^{n-1} (d_{k-1}^2 + \sum_{i=3}^{k-2} d_i^1 \beta_{k-i-2}^2) e_k \\ &\quad +(d_{n-1}^2 + d_2^1\theta_3 - d_{n-1}^1\alpha + \sum_{i=3}^{n-2} d_i^1 \beta_{n-i-2}^2) e_n \,. \end{aligned}$$

Finally,

$$\begin{aligned} d(e_3) &= (d_2^2 + d_1^1)e_3 + d_3^2 e_4 + \textstyle\sum_{k=5}^{n-1} (d_{k-1}^2 + \sum_{i=3}^{k-2} d_i^1 \beta_{k-i-2}^2) e_k \quad (3) \\ &\quad +(d_{n-1}^2 + d_2^1\theta_3 - d_{n-1}^1\alpha + \textstyle\sum_{i=3}^{n-2} d_i^1 \beta_{n-i-2}^2) e_n \end{aligned}$$

Now, we will compute $d(e_4)$ using the same method like $d(e_3)$. Since $[e_3, e_1] = e_4$ then

$$\begin{aligned} d(e_4) &= [d(e_3), e_1] + [e_3, d(e_1)] \\ &= [\textstyle\sum_{i=3}^{n} d_i^3 e_i, e_1] + [e_3, \sum_{i=1}^{n} d_i^1 e_i] \\ &= [\textstyle\sum_{i=3}^{n-1} d_i^3 e_i, e_1] + d_n^3[e_n, e_1] + d_1^1[e_3, e_1] + [e_3, \sum_{i=2}^{n} d_i^1 e_i] \\ &= \sum_{i=3}^{n-1} d_i^3 e_{i+1} + d_n^3[e_n, e_1] + d_1^1[e_3, e_1] + \sum_{i=2}^{n-3} d_i^1[e_3, e_i] \\ &\quad + d_{n-2}^1[e_3, e_{n-2}] + d_{n-1}^1[e_3, e_{n-1}] + d_n^1[e_3, e_n] \end{aligned}$$

Using Theorem 2.1 we have:

$$\begin{cases} [\sum_{i=3}^{n} d_i^3 e_i, e_1] = \sum_{i=3}^{n-1} d_i^3 e_{i+1} \\ [e_3, e_1] = e_4 \\ [e_3, e_{n-2}] = \alpha(-1)^4 e_n = \alpha e_n \\ [e_3, e_{n-1}] = 0 \\ [e_3, e_n] = 0. \end{cases}$$

Therefore

$$\begin{aligned} d(e_4) &= \textstyle\sum_{i=4}^{n-1} d_{i-1}^3 e_i + d_{n-1}^3 e_n + d_1^1[e_3, e_1] + \sum_{i=2}^{n-3} d_i^1[e_3, e_i] + d_{n-2}^1 \alpha e_n \\ &= d_1^1 e_4 + \textstyle\sum_{i=4}^{n-1} d_{i-1}^3 e_i + d_{n-1}^3 e_n + \sum_{i=2}^{n-3} d_i^1[e_3, e_i] + d_{n-2}^1 \alpha e_n \end{aligned}$$

According to Theorem 2.1

$[e_3, e_i] \in$ span $(e_{i+3}, e_{i+4}, \dots, e_n)$ therefore $\exists\ \beta_{k-i-3}^3 \in \mathbb{C}, k = i+3, i+4, \dots, n$, such that $[e_3, e_i] = \sum_{k=i+3}^{n} \beta_{k-i-3}^3 e_k$.

Now we have

$$\begin{aligned} \textstyle\sum_{i=2}^{n-3} d_i^1[e_3,e_i] &= \textstyle\sum_{i=2}^{n-3} d_i^1 \sum_{k=i+3}^{n} \beta_{k-i-3}^3 e_k \\ &= \textstyle\sum_{i=2}^{n-3} \sum_{k=i+3}^{n} d_i^1 \beta_{k-i-3}^3 e_k \end{aligned}$$

To invert the sums of double summation, we use:

$$\begin{cases} 2 \leqslant i \leqslant n-3 \\ i+3 \leqslant k \leqslant n \end{cases} \Leftrightarrow \begin{cases} 5 \leqslant k \leqslant n \\ 2 \leqslant i \leqslant k-3 \end{cases} \qquad (4)$$

Now, we get:

$$\textstyle\sum_{i=2}^{n-3} d_i^1[e_3,e_i] = \sum_{k=5}^{n} (\sum_{i=2}^{k-3} d_i^1 \beta_{k-i-3}^3) e_k$$

summing up

$$\begin{aligned} d(e_4) &= d_1^1 e_4 + \textstyle\sum_{i=4}^{n-1} d_{i-1}^3 e_i + d_{n-1}^3 e_n + \sum_{i=2}^{n-2} d_i^1[e_3, e_i] + d_{n-2}^1 \alpha e_n \\ &= d_1^1 e_4 + \textstyle\sum_{i=4}^{n-1} d_{i-1}^3 e_i + \sum_{k=5}^{n} (\sum_{i=2}^{k-3} d_i^1 \beta_{k-i-3}^3) e_k + d_{n-2}^1 \alpha e_n \\ &= (d_1^1 + d_3^3) e_4 + \textstyle\sum_{k=5}^{n-1} d_{k-1}^3 e_k + \sum_{k=5}^{n-1} (\sum_{i=2}^{k-3} d_i^1 \beta_{k-i-3}^3) e_k + d_{n-2}^1 \alpha e_n \\ &\quad +(d_{n-1}^3 + d_{n-2}^1 \alpha + \textstyle\sum_{i=2}^{n-3} d_i^1 \beta_{n-i-3}^3) e_n \,. \end{aligned}$$

Finally,

$$\begin{aligned} d(e_4) &= (d_1^1 + d_3^3) e_4 + \textstyle\sum_{k=5}^{n-1} d_{k-1}^3 e_k + \sum_{k=5}^{n-1} (\sum_{i=2}^{k-3} d_i^1 \beta_{k-i-3}^3) e_k \qquad (5) \\ &\quad +(d_{n-1}^3 + d_{n-2}^1 \alpha + \textstyle\sum_{i=2}^{n-3} d_n^1 \beta_{n-i-3}^3) e_n \end{aligned}$$

Now, we would like to compute $d(e_5)$. We restart the same computation like $d(e_4)$ and $d(e_3)$. We consider $[e_4, e_1] = e_5$, then

$$\begin{aligned} d(e_5) &= [d(e_4), e_1] + [e_4, d(e_1)] \\ &= [\textstyle\sum_{i=4}^{n} d_i^4 e_i, e_1] + [e_4, \sum_{i=1}^{n} d_i^1 e_i] \end{aligned}$$

$= [\sum_{i=4}^{n-1} d_i^4 e_i, e_1] + d_n^4[e_n, e_1] + d_1^1[e_4, e_1] + [e_4, \sum_{i=2}^{n} d_i^1 e_i]$

$= \sum_{i=4}^{n-1} d_i^4 e_{i+1} + d_1^1[e_4, e_1] + \sum_{i=2}^{n-4} d_i^1[e_4, e_i] + d_{n-3}^1[e_4, e_{n-3}]$

$+d_{n-2}^1[e_4, e_{n-2}] + d_{n-1}^1[e_4, e_{n-1}] + d_n^1[e_4, e_n]$

Using Theorem 2.1 we have

$$\begin{cases} [\sum_{i=4}^{n} d_i^4 e_i, e_1] = \sum_{i=4}^{n-1} d_i^4 e_{i+1} \\ [e_4, e_1] = e_5 \\ [e_4, e_{n-3}] = \alpha(-1)^5 e_n = -\alpha e_n \\ [e_4, e_{n-2}] = 0 \\ [e_4, e_{n-1}] = 0 \\ [e_4, e_n] = 0 . \end{cases}$$

We get

$d(e_5) = \sum_{i=4}^{n-1} d_i^4 e_{i+1} + d_1^1[e_4, e_1] + \sum_{i=2}^{n-4} d_i^1[e_4, e_i] - d_{n-3}^1 \alpha e_n$

$= \sum_{i=5}^{n-1} d_{i-1}^4 e_i + d_{n-1}^4 e_n + d_1^1[e_4, e_1] + \sum_{i=2}^{n-4} d_i^1[e_4, e_i] - d_{n-3}^1 \alpha e_n$

$= d_1^1 e_5 + \sum_{i=5}^{n-1} d_{i-1}^4 e_i + d_{n-1}^4 e_n + \sum_{i=2}^{n-4} d_i^1[e_4, e_i] - d_{n-3}^1 \alpha e_n$

Using the Theorem 2.1 we get:

Since, $[e_4, e_i] \in$ span $(e_{i+4}, e_{i+5}, \dots, e_n)$ then $\exists\ \beta_{k-i-4}^4 \in \mathbb{C}, k = i+4, i+5, \dots, n$ such that $[e_4, e_i] = \sum_{k=i+4}^{n} \beta_{k-i-4}^4 e_k$.

Now we have

$$\sum_{i=2}^{n-4} d_i^1[e_4, e_i] = \sum_{i=2}^{n-4} d_i^1 \sum_{k=i+4}^{n} \beta_{k-i-4}^4 e_k = \sum_{i=2}^{n-4} \sum_{k=i+4}^{n} d_i^1 \beta_{k-i-4}^4 e_k$$

We can invert the sums in double summation, using

$$\begin{cases} 2 \leqslant i \leqslant n-4 \\ i+4 \leqslant k \leqslant n \end{cases} \Leftrightarrow \begin{cases} 6 \leqslant k \leqslant n \\ 2 \leqslant i \leqslant k-4 \end{cases} \quad (6)$$

Now, we get:

$$\sum_{i=2}^{n-4} d_i^1[e_4, e_i] = \sum_{k=6}^{n} (\sum_{i=2}^{k-4} d_i^1 \beta_{k-i-4}^4) e_k$$

Summing up:

$d(e_5) = d_1^1 e_5 + \sum_{i=5}^{n-1} d_{i-1}^4 e_i + d_{n-1}^4 e_n + \sum_{i=2}^{n-2} d_i^1[e_4, e_i] - d_{n-3}^1 \alpha e_n$

$= (d_1^1 + d_4^4) e_5 + \sum_{i=6}^{n-1} d_{i-1}^4 e_i + d_{n-1}^4 e_n + \sum_{k=6}^{n} (\sum_{i=2}^{k-4} d_i^1 \beta_{k-i-4}^4) e_k - d_{n-3}^1 \alpha e_n$

$= (d_1^1 + d_4^4) e_5 + \sum_{i=6}^{n-1} d_{i-1}^4 e_i + \sum_{k=6}^{n-1} (\sum_{i=2}^{k-4} d_i^1 \beta_{k-i-4}^4) e_k$

$+(d_{n-1}^4 - d_{n-3}^1 \alpha + \sum_{i=2}^{n-4} d_i^1 \beta_{n-i-4}^4) e_n$.

Finally,

$$d(e_5) = (d_1^1 + d_4^4) e_5 + \sum_{i=6}^{n-1} d_{i-1}^4 e_i + \sum_{k=6}^{n-1} (\sum_{i=2}^{k-4} d_i^1 \beta_{k-i-4}^4) e_k + (d_{n-1}^4 - d_{n-3}^1 \alpha + \sum_{i=2}^{n-4} d_n^1 \beta_{n-i-4}^4) e_n \quad (7)$$

A generalization of previous calculations can be written and a general formula can be given as follows:

$$d(e_\lambda) = (d_1^1 + d_{\lambda-1}^{\lambda-1}) e_\lambda + \sum_{k=\lambda+1}^{n-1} (d_{k-1}^{\lambda-1} - \sum_{i=2}^{k-\lambda} d_i^1 \beta_{k-i-\lambda+1}^{\lambda-1}) e_k + (d_{n-1}^{\lambda-1} + (-1)^\lambda \alpha d_{n-\lambda+2}^1 - \sum_{i=2}^{n-\lambda} d_n^1 \beta_{n-i-\lambda+1}^{\lambda-1}) e_n . \quad (8)$$

It is clear that this relation is true for $\lambda \geqslant 4$. We consider, $e_n = [e_{n-1}, e_1]$ then $d(e_n) = d([e_{n-1}, e_1])$. So,

$d(e_n) = [d(e_{n-1}), e_1] + [e_{n-1}, d(e_1)]$

$= [\sum_{i=n-1}^{n} d_i^{n-1} e_i), e_1] + [e_{n-1}, \sum_{i=1}^{n} d_i^1 e_i]$

$= d_{n-1}^{n-1}[e_{n-1}, e_1] + d_n^{n-1}[e_n, e_1] + d_1^1[e_{n-1}, e_1] + d_2^1[e_{n-1}, e_2]$

$+[e_{n-1}, \sum_{i=3}^{n} d_i^1 e_i]$

Due to Theorem 2.1 we have

$$\begin{cases} [e_{n-1}, e_1] = e_n \\ [e_n, e_1] = 0 \\ [e_{n-1}, e_2] = \alpha e_n \\ [e_{n-1}, \sum_{i=3}^{n} d_i^1 e_i] = 0 . \end{cases}$$

Finally, we conclude that

$$d(e_n) = (d_{n-1}^{n-1} + d_1^1 + \alpha d_2^1) e_n \quad (9)$$

Now (1) and (9) give

$$d_{n-1}^{n-1} = d_1^1 + d_2^1(\tfrac{\theta_2}{\theta_1} - \alpha) \quad (10)$$

On the other hand,

$$[e_2, e_2] = \theta_3 e_n$$

Then

$d(e_n) = \frac{1}{\theta_3}([d(e_2), e_2] + [e_2, d(e_2)]))$

$= \frac{1}{\theta_3}([\sum_{i=2}^{n} d_i^2 e_i, e_2] + [e_2, \sum_{i=2}^{n} d_i^2 e_i])$

$= \frac{1}{\theta_3}(d_2^2[e_2, e_2] + [\sum_{i=3}^{n} d_i^2 e_i, e_2] + d_2^2[e_2, e_2] + [e_2, \sum_{i=3}^{n} d_i^2 e_i])$

$= \frac{1}{\theta_3}(2 d_2^2[e_2, e_2])$

$= \frac{1}{\theta_3}(2 d_2^2 \theta_3) e_n$

Therefore

$$d(e_n) = 2 d_2^2 e_n . \quad (11)$$

Comparing (1) and (11) we obtain

$$d_2^2 = d_1^1 + d_2^1(\tfrac{\theta_2}{2\theta_1}) \quad (12)$$

Also, using $[e_{n-1}, e_2] = \alpha e_n$ we find

$d(e_n) = \frac{1}{\alpha}([d(e_{n-1}), e_2] + [e_{n-1}, d(e_2)])$

$= \frac{1}{\alpha}([\sum_{i=n-1}^{n} d_i^{n-1}(e_i), e_2] + [e_{n-1}, \sum_{i=2}^{n} d_i^2 e_i])$

$= \frac{1}{\alpha}(d_{n-1}^{n-1}[e_{n-1}, e_2] + d_n^{n-1}[e_n, e_2] + d_2^2[e_{n-1}, e_2]$

$+[e_{n-1}, \sum_{i=3}^{n} d_i^2 e_i])$

$= \frac{1}{\alpha}((d_{n-1}^{n-1} + d_2^2)[e_{n-1}, e_2] + d_n^{n-1}[e_n, e_2]$

$+[e_{n-1}, \sum_{i=3}^{n} d_i^2 e_i])$

$= \frac{1}{\alpha}((d_{n-1}^{n-1} + d_2^2)[e_{n-1}, e_2] + d_n^{n-1}[e_n, e_2]$

$+[e_{n-1}, \sum_{i=3}^{n} d_i^2 e_i])$

$$d(e_n) = (d_2^2 + d_{n-1}^{n-1}) e_n \quad (13)$$

Now(1) and (13) give

$$d_{n-1}^{n-1} = 2 d_1^1 + d_2^1(\tfrac{\theta_2}{\theta_1}) - d_2^2 \quad (14)$$

From (10) and (14) we get

$$d_2^2 = d_2^1 \alpha + d_1^1 . \quad (15)$$

But, from (12) and (15) we find

$$d_2^1 = 0 . \quad (16)$$

Again due to Theorem 2.1 , $[e_1, e_2] = -e_3 + \theta_2 e_n$ then

$d(e_3) = -([d(e_1), e_2] + [e_1, d(e_2)] - \theta_2 d(e_n))$

$= -([\sum_{i=1}^{n} d_i^1 e_i, e_2] + [e_1, \sum_{i=2}^{n} d_i^2 e_i] - \theta_2 d(e_n))$

$= -(d_1^1[e_1, e_2] + [\sum_{i=2}^{n} d_i^1 e_i, e_2] + d_2^2[e_1, e_2] + [e_1, \sum_{i=3}^{n} d_i^2 e_i] - 2 d_2^2 \theta_2 e_n)$

$= -(d_1^1[e_1, e_2] + [\sum_{i=2}^{n-2} d_i^1 e_i, e_2] + d_{n-1}^1[e_{n-1}, e_2] + d_n^1[e_n, e_2] + d_2^2[e_1, e_2]$

$+[e_1, \sum_{i=3}^{n-1} d_i^2 e_i] + d_n^2[e_1, e_n] - 2d_2^2\theta_2 e_n)$
$= -((d_1^1 + d_2^2)[e_1, e_2] + [\sum_{i=2}^{n-2} d_i^1 e_i, e_2] + \alpha d_{n-1} e_n$
$+[e_1, \sum_{i=3}^{n-1} d_i^2 e_i] - 2d_2^2\theta_2 e_n)$
$= -((d_1^1 + d_2^2)[e_1, e_2] + [\sum_{i=2}^{n-2} d_i^1 e_i, e_2] + \alpha d_{n-1}^1 e_n$
$-\sum_{i=3}^{n-1} d_i^2 e_{i+1} - 2d_2^2\theta_2 e_n)$
$= -((d_1^1 + d_2^2)[e_1, e_2] + \sum_{i=2}^{n-2} d_i^1[e_i, e_2] + \alpha d_{n-1}^1 e_n$
$-\sum_{i=3}^{n-1} d_i^2 e_{i+1} - 2d_2^2\theta_2 e_n)$

Again dule to Theorem 2.1, $[e_i, e_2] \in$ span $(e_{i+2}, e_{i+3}, \dots, e_n)$ hence $\exists\ \beta_{k-i-2}^2, k = i+2, i+3, \dots, n$ such that $[e_i, e_2] = \sum_{k=i+2}^{n} \beta_{k-i-2}^2 e_k$. Therefore we have

$$\sum_{i=2}^{n-2} d_i^1[e_i, e_2] = \sum_{i=2}^{n-2} d_i^1 \sum_{k=i-2}^{n} \beta_{k-i-2}^2 e_k = \sum_{i=2}^{n-2} \sum_{k=i+2}^{n} d_i^1 \beta_{k-i-2}^2 e_k$$

We can invert the sums in double summation, as follows

$$\begin{cases} 2 \leqslant i \leqslant n-2 \\ i+2 \leqslant k \leqslant n \end{cases} \Leftrightarrow \begin{cases} 4 \leqslant k \leqslant n \\ 2 \leqslant i \leqslant k-2 \end{cases} \qquad (17)$$

Hence, we get:

$$\sum_{i=2}^{n-2} d_i^1[e_i, e_2] = \sum_{k=4}^{n} (\sum_{i=2}^{k-2} d_i^1 \beta_{k-i-2}^2) e_k$$

Thus,

$$\begin{aligned} d(e_3) &= -((d_1^1 + d_2^2)[e_1, e_2] + \sum_{k=4}^{n} (\sum_{i=2}^{k-2} d_i^1 \beta_{k-i-2}^2) e_k + \alpha d_{n-1}^1 e_n \\ &\quad - \sum_{i=3}^{n-1} d_i^2 e_{i+1} - 2d_2^2\theta_2 e_n) \\ &= -(-(d_1^1 + d_2^2)e_3 + (d_1^1 + d_2^2)\theta_2 e_n + \sum_{k=4}^{n} (\sum_{i=2}^{k-2} d_i^1 \beta_{k-i-2}^2) e_k + \alpha d_{n-1}^1 e_n \\ &\quad - \sum_{i=3}^{n-1} d_i^2 e_{i+1} - 2d_2^2\theta_2 e_n) \\ &= ((d_1^1 + d_2^2)e_3 - (d_1^1 + d_2^2)\theta_2 e_n - \sum_{k=4}^{n} (\sum_{i=2}^{k-2} d_i^1 \beta_{k-i-2}^2) e_k - \alpha d_{n-1}^1 e_n \\ &\quad + \sum_{i=3}^{n-1} d_i^2 e_{i+1} + 2d_2^2\theta_2 e_n) \\ &= ((d_1^1 + d_2^2)e_3 - \sum_{k=4}^{n-1} (\sum_{i=2}^{k-2} d_i^1 \beta_{k-i-2}^2) e_k + \sum_{i=4}^{n-1} d_{i-1}^2 e_i) \\ &\quad +(2d_2^2\theta_2 - \alpha d_{n-1}^1 - (d_1^1 + d_2^2)\theta_2 + d_{n-1}^2 - \sum_{i=2}^{n-2} d_i^1 \beta_{n-i-2}^2) e_n \\ &= (d_1^1 + d_2^2)e_3 + \sum_{i=4}^{n-1} d_{i-1}^2 e_i - \sum_{k=4}^{n-1} (\sum_{i=2}^{k-2} d_i^1 \beta_{k-i-2}^2) e_k \\ &\quad +(2d_2^2\theta_2 - \alpha d_{n-1}^1 - (d_1^1 + d_2^2)\theta_2 + d_{n-1}^2 - \sum_{i=2}^{n-2} d_i^1 \beta_{n-i-2}^2) e_n \\ &= (d_1^1 + d_2^2)e_3 + \sum_{k=4}^{n-1} d_{k-1}^2 e_k - \sum_{k=4}^{n-1} (\sum_{i=2}^{k-2} d_i^1 \beta_{k-i-2}^2) e_k \\ &\quad +(2d_2^2\theta_2 - \alpha d_{n-1}^1 - (d_1^1 + d_2^2)\theta_2 + d_{n-1}^2 - \sum_{i=2}^{n-2} d_i^1 \beta_{n-i-2}^2) e_n \end{aligned}$$

$$d(e_3) = (d_1^1 + d_2^2)e_3 + \sum_{k=4}^{n-1} d_{k-1}^2 e_k - \sum_{k=4}^{n-1} (\sum_{i=2}^{k-2} d_i^1 \beta_{k-i-2}^2) e_k + (2d_2^2\theta_2 - \alpha d_{n-1}^1 - (d_1^1 + d_2^2)\theta_2 + d_{n-1}^2 - \sum_{i=2}^{n-2} d_i^1 \beta_{n-i-2}^2) e_n . \qquad (18)$$

Using (3) and (18) we conclude that

$$\begin{cases} d_2^1 = 0, \\ d_i^1 = 0, i = 3,4,5,\dots,n-3 \\ d_{n-2}^1 = 0 . \end{cases} \qquad (19)$$

The matrix of d has the form $D = (d_k^l)_{k,l=1,2,3,\dots,n}$ where

$$\begin{cases} d_2^1 = 0, \\ d_i^1 = 0, i = 3,4,5,\dots,n-2 \\ d_2^2 = d_1^1 . \end{cases} \qquad (20)$$

Finally, $\dim Der(L) = n+1$. □

3. Conclusions

In this paper, we describe and generalized all derivations of third class of filiform Leibniz algebra. All derivations Fended between $n+1$ and $2n-1$, where n is number of dimension.

The pervious are studied derivations in Third class of filiform Leibniz algebra in low dimension from five to eight dimension only, but in this paper, we generalized for any dimension.

In theorem (2.1) describe third class of Filiform Leibniz algebra in (n+1)-dimensional, and theorem (2.2) describe the derivation algebras which elements from TLb_n.

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